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INTEGRAL EQUATIONS OF ELASTICITY THEORY FOR A MULTICONNECTED DOMAIN WITH INCLUSIONS*

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An infinite multiconnected domain that contains curvilinear cracks, holes arbitrary inclusions, and also rectilinear bonding stringers, is examined. The defect and foreign body geometry and arrangement are arbitrary, but it is assumed that they do not intersect. Different special cases of crack, hole, stringer, and inclusion interaction have been studied in /1-18/. A system of equations describing the state of stress of such a composite medium is constructed in a general formulation. The problem is solved by using the introduction of complex potentials and the theory of singular integral equations (SIE) /19, 20/. Numerical realization of the SIE is achieved by using interpolation formulas and the Lobatto-Chebyshev method.

1. Formulation of the problem. We consider a plane multiconnected domain referred to a Cartesian xy system of coordinates stretched (or compressed) at infinity by mutually perpendicular forces of strengths N_1 and N_2. The stress N_1 makes an angle α with the x axis (Fig.1). The composite medium is in equilibrium under the action of selfequilibrated forces applied to the crack edges and the hole outlines, and of the contact stresses occurring on the boundaries between the isotropic inclusions and the isotropic infinite plane (the inclusions and stringers can be from different elastic materials).

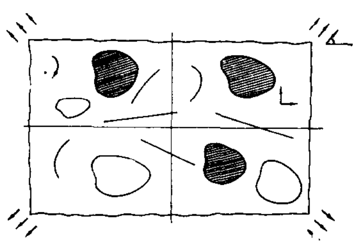

Fig.1

The composite medium (Fig.1) consists of an infinite isotropic plane S in which there are M_1 curvilinear cracks, M_2 curvilinear closed holes, M_3 rectilinear stringers, and M_4 curvilinear elastic inclusions. Besides the given forces, concentrated forces P_j+iQ_j at z_j^* $(j=1,\ldots,K^*)$ and moments M_j at z_j^{**} $(j=1,\ldots,K^{**})$ can act in the plane of the medium.

The following assignment of the boundary conditions is possible.

Normal and tangential stresses (+ for the upper edge and − for the lower edge)

$$(\sigma_n^\pm - i\sigma_t^\pm)\,|_{l_j} \tag{1.1}$$

are given on the crack edges l_j $(l_j=1,\ldots,M_1)$

The forces

$$(\sigma_n - i\sigma_t)\,|_{\gamma_j^*} \tag{1.2}$$

are given on the hole outlines γ_j^* $(j=1,\ldots,M_2)$.

The following system of boundary conditions /14, 15/

$$\sigma_n^+=\sigma_n^-,\quad \varepsilon_0=du_t^+/dt=du_t^-/dt,\quad u_n^++iu_t^+=u_n^-+iu_t^- \tag{1.3}$$

are given on the boundary L_j $(j=1,\ldots M_3)$ between the infinite medium and the rectilinear elastic inclusion (stringer).

The first two conditions (1.3) result in the expression

$$ih\,[(\sigma_n^+-i\sigma_t^+)-(\sigma_n^--i\sigma_t^-)\;+\frac{E^{(j)}S^{(j)}}{E}\frac{d}{dt}[(\sigma_n^++\sigma_s^+)-(1+\nu)\,\sigma_n^+]=0 \tag{1.4}$$

where $(\sigma_n^+, \sigma_s^+, \sigma_t^+)$ are contact stresses occurring on the boundary L_j between the infinite plate and the stringer L_j, h is the plate thickness, $E^{(j)}$, E are the elastic moduli of the stringer L_j and the plate S, respectively, ν is Poisson's ratio, and $S^{(j)}$ is the cross-sectional area of the stringer L_j.

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It is known that the contact stresses on the contact boundary of the inclusions γ_j $(j = 1, \ldots, M_4)$ and the medium S are mutually equal, i.e.

$$(\sigma_n^+ - i\sigma_t^+)|_{\gamma_j} = (\sigma_n^- - i\sigma_t^-)|_S \tag{1.5}$$

and the displacement jump is given by

$$g_0^{(j)}(t) = 2\mu_j\left[-\frac{d}{dt}(u_j^+(t) - u_j^-(t)) + i\frac{d}{dt}(v_j^+(t) - v_j^-(t))\right] \tag{1.6}$$

where μ_j is the shear modulus of the inclusions γ_j, u_j and v_j are displacement components along the x and y axes in the general system of coordinates (unlike u_n and u_t, v_n and v_t).

2. *Construction of the complex potentials.* The Kolosov-Muskhelishvili potentials /19/ for the general case under consideration have the following form

$$\Phi_0(z) = \Phi(z) + \Gamma - \sum_{j=1}^{K^*}\frac{P_j + iQ_j}{2\pi(1+k)}\frac{1}{z - z_j^*} \tag{2.1}$$

$$\Psi_0(z) = \Psi(z) + \Gamma' + \sum_{j=1}^{K^*}\left[\frac{k(P_j - iQ_j)}{2\pi(1+k)}\frac{1}{z - z_j^*} - \frac{\bar{z}_j^*(P_j + iQ_j)}{2\pi(1+k)} \times \frac{1}{(z - z_j^*)^2}\right] - i\sum_{j=1}^{K^{**}}\frac{M_j}{2\pi}\frac{1}{(z - z_j^{**})^2} \tag{2.2}$$

Here (ε_∞ is the rotation at infinity)

$$\Gamma = \frac{1}{4}(N_1 + N_2) + \frac{2i\mu\varepsilon_\infty}{1+k}, \quad \Gamma' = -\frac{1}{2}(N_1 - N_2)e^{-2i\alpha}$$

$$\Phi(z) = \sum_{j=1}^{M_1}\frac{1}{2\pi i}\int_{l_j}\frac{\varphi_j(\tau)}{\tau - z}d\tau + \sum_{j=1}^{M_2}\frac{1}{2\pi i}\oint_{\gamma_j^*}\frac{\varphi_j^*(\tau)}{\tau - z}d\tau + \sum_{j=1}^{M_3}\frac{1}{2\pi i}\int_{L_j}\frac{\mu_j(\tau)}{\tau - z}d\tau + \sum_{j=1}^{M_4}\frac{1}{2\pi i}\oint_{\gamma_j}\frac{G_j(\tau)}{\tau - z}d\tau \tag{2.3}$$

$$k = \begin{cases} 3 - 4\nu & \text{for plane strain,} \\ (3-\nu)/(1+\nu) & \text{for the plane state of stress,} \end{cases}$$

and $\{\varphi_j(t), \varphi_j^*(t), \mu_j(t), G_j(t)\}$ are unknown densities on the boundaries $\{l_j, \gamma_j^*, L_j, \gamma_j\}$ respectively.

To construct an expression for the potential $\Psi(z)$ as a function of the densities $\{\varphi_j(t), \varphi_j^*(t), \mu_j(t), G_j(t)\}$, we write the boundary conditions of the problem by using the complex potentials $\Phi_0(z)$ and $\Psi_0(z)$.

The normal and tangential forces on the crack edges l_j $(j = 1, \ldots, M_1)$ have the form

$$(\sigma_n^\pm - i\sigma_t^\pm)|_{l_j} = \Phi_0^\pm(t) + \overline{\Phi_0^\pm(t)} + \frac{dt}{dt}[\bar{t}\Phi_0'^\pm(t) + \Psi_0^\pm(t)] \quad t \in l_j. \tag{2.4}$$

An analogous expression

$$q_j^*(t) = \Phi_0(t) + \overline{\Phi_0(t)} + \frac{dt}{dt}[\bar{t}\Phi_0'(t) + \Psi_0(t)], \quad t \in \gamma_j^* \tag{2.5}$$

holds on the closed contour γ_j^* $(j = 1, \ldots, M_2)$ where $q_j^*(t) = (\sigma_n - i\sigma_t)|_{\gamma_j^*}$ are given forces on the contour γ_j^*.

The following two conditions will be satisfied on L_j $(j = 1, \ldots, M_3)$ (the third equality of (1.3) and (1.4) is taken into account):

$$\Phi_0^+(t) - k\overline{\Phi_0^+(t)} + \frac{dt}{dt}[\bar{t}\Phi_0'^+(t) + \Psi_0^+(t)] = \Phi_0^-(t) - k\overline{\Phi_0^-(t)} + \frac{dt}{dt}[\bar{t}\Phi_0'^-(t) + \Psi_0^-(t)], \quad t \in L_j \tag{2.6}$$

$$ih\left[(\Phi_0^+(t)-\Phi_0^-(t))+(\overline{\Phi_0^+(t)}-\overline{\Phi_0^-(t)})+\frac{dt}{d\bar{t}}[\bar{t}(\Phi_0'^+(t)-\Phi_0'^-(t))+(\Psi_0^+(t)-\Psi_0^-(t))]\right]+\frac{E^{(j)}S^{(j)}}{E}\frac{d}{dt}\left[2(\Phi_0^+(t)+\overline{\Phi_0^+(t)}-(1+\nu)\left\{\Phi_0^+(t)+\overline{\Phi_0^+(t)}+\operatorname{Re}\frac{dt}{d\bar{t}}[\bar{t}\Phi_0'^+(t)+\Psi_0^+(t)]\right\}\right]=0,\quad t\in L_j \tag{2.7}$$

Finally, conditions (1.5) and (1.6) on the contact boundaries γ_j $(j=1,\ldots,M_4)$ between the inclusions and the infinite medium are written as follows

$$\Phi_0^+(t)+\overline{\Phi_0^+(t)}+\frac{dt}{d\bar{t}}[\bar{t}\Phi_0'^+(t)+\Psi_0^+(t)]=\Phi_0^-(t)+\overline{\Phi_0^-(t)}+\frac{dt}{d\bar{t}}[\bar{t}\Phi_0'^-(t)+\Psi_0^-(t)],\quad t\in\gamma_j \tag{2.8}$$

$$\Phi_0^+(t)-k_j\overline{\Phi_0^+(t)}+\frac{dt}{d\bar{t}}[\bar{t}\Phi_0'^+(t)+\Psi_0^+(t)]=\Gamma_0^{(j)}\left\{\Phi_0^-(t)-k\overline{\Phi_0^-(t)}+\frac{dt}{d\bar{t}}[\bar{t}\Phi_0'^-(t)+\Psi_0^-(t)]\right\}+g_0^{(j)}(t)$$
$$t\in\gamma_j\quad(\Gamma_0^{(j)}=\mu_j/\mu) \tag{2.9}$$

Solving each of expressions (2.4)-(2.6) and (2.8) for $\Psi^+(t)-\Psi^-(t)$ and applying the known Sokhotskii-Plemelj formula, we obtain the expression for $\Psi(z)$:

$$\Psi(z)=\sum_{j=1}^{M_1}\left[\frac{1}{2\pi i}\int_{l_j}\frac{q_1^{(j)}(\tau)}{\tau-z}\overline{d\tau}-\frac{1}{2\pi i}\int_{l_j}\frac{\overline{\varphi_j(\tau)}}{\tau-z}\overline{d\tau}-\frac{1}{2\pi i}\int_{l_j}\frac{\bar{\tau}\varphi_j(\tau)}{(\tau-z)^2}d\tau\right]+\sum_{j=1}^{M_2}\left[\frac{1}{2\pi i}\oint_{\gamma_j^*}\frac{q_j^*(\tau)}{\tau-z}\overline{d\tau}-\frac{1}{2\pi i}\oint_{\gamma_j^*}\frac{\overline{\varphi_j^*(\tau)}}{\tau-z}\overline{d\tau}-\frac{1}{2\pi i}\oint_{\gamma_j^*}\frac{\bar{\tau}\varphi_j^*(\tau)}{(\tau-z)^2}d\tau\right]+\sum_{j=1}^{M_3}\left[\frac{k}{2\pi i}\int_{L_j}\frac{\overline{\mu_j(\tau)}}{\tau-z}\overline{d\tau}-\frac{1}{2\pi i}\int_{L_j}\frac{\bar{\tau}\mu_j(\tau)}{(\tau-z)^2}d\tau\right]+\sum_{j=1}^{M_4}\left[-\frac{1}{2\pi i}\oint_{\gamma_j}\frac{\overline{G_j(\tau)}}{\tau-z}\overline{d\tau}-\frac{1}{2\pi i}\oint_{\gamma_j}\frac{\bar{\tau}G_j(\tau)}{(\tau-z)^2}d\tau\right] \tag{2.10}$$

Here

$$q_1^{(j)}(t)=[\sigma_n^{(j)+}-\sigma_n^{(j)-}]-i[\sigma_t^{(j)+}-\sigma_t^{(j)-}],\quad t\in l_j\quad(j=1,\ldots,M_1)$$
$$q_j^*(t)=\sigma_n^{(j)}-i\sigma_t^{(j)},\quad t\in\gamma_j^*\quad(j=1,\ldots,M_2)$$

3. ***Derivation of the system of SIE.*** We will now set up the system of equations satisfying the boundary conditions. Taking account of expression (2.4) and the Sokhotskii-Plemelj formula for $\Phi_0(z)$ and $\Psi_0(z)$ as $z\to t\in l_k$, we obtain the following system of SIE

$$\frac{1}{\pi i}\int_{l_k}\frac{\varphi_k(\tau)}{\tau-t}d\tau-\frac{1}{\pi i}\int_{l_k}\frac{\overline{\varphi_k(\tau)}}{\bar{\tau}-\bar{t}}\overline{d\tau}-\frac{dt}{d\bar{t}}\left[\frac{1}{\pi i}\int_{l_k}\frac{\overline{\varphi_k(\tau)}}{\tau-t}\overline{d\tau}+\frac{1}{\pi i}\int_{l_k}\frac{\bar{\tau}-\bar{t}}{(\tau-t)^2}\varphi_k(\tau)d\tau\right]+\sum_{\substack{j=1\\j\neq k}}^{M}\frac{1}{\pi i}\int_{L_j^*}[K_{11}^{(j)}(\tau,t)\nu_j(\tau)+K_{12}^{(j)}(\tau,t)A(\tau,\bar{\tau})\overline{\nu_j(\tau)}]d\tau=$$
$$A_1^{(k)}(t,\bar{t})-\frac{dt}{d\bar{t}}\left\{\sum_{j=1}^{M_1}\frac{1}{\pi i}\int_{l_j}\frac{q_1^{(j)}(\tau)}{\tau-t}\overline{d\tau}+\frac{1}{2\pi i}\oint_{\gamma_j^*}\frac{q_j^*(\tau)}{\tau-t}\overline{d\tau}\right\}$$
$$t\in l_k\quad(k=1,\ldots,M_1) \tag{3.1}$$

Here

$$A(\tau, \bar{\tau}) = d\bar{\tau}/d\tau,$$

$$K_{11}^{(j)}(\tau, t) = \frac{1}{\tau - t}\left(1 - \frac{dt}{d\bar{t}}\frac{\bar{\tau} - \bar{t}}{\tau - t}\right), \quad j = 1, \ldots, M$$

$$K_{12}^{(j)}(\tau, t) = \begin{cases} -\dfrac{1}{\bar{\tau} - \bar{t}}\left(1 + \dfrac{dt}{d\bar{t}}\dfrac{\bar{\tau} - \bar{t}}{\tau - t}\right), & j = 1, \ldots, M_{12}, M_{123} + 1, \ldots, M \\ -\dfrac{1}{\bar{\tau} - \bar{t}}\left(1 - k\dfrac{dt}{d\bar{t}}\dfrac{\bar{\tau} - \bar{t}}{\tau - t}\right), & j = M_{12} + 1, \ldots, M_{123} \end{cases}$$

$$M_1 + M_2 = M_{12}, \; M_{12} + M_3 = M_{123}, \; M_{123} + M_4 = M$$

$$L_j^* \to l_j, \; \nu_j(t) = \varphi_j(t), \; j = 1, \ldots, M_1$$

$$L_{j+M_1}^* \to \gamma_j^*, \; \nu_{j+M_1}(t) = \varphi_j^*(t), \; j = 1, \ldots, M_2$$

$$L_{j+M_{12}}^* \to L_j, \; \nu_{j+M_{12}}(t) = \mu_j(t), \; j = 1, \ldots, M_3$$

$$L_{j+M_{123}}^* \to \gamma_j, \; \nu_{j+M_{123}}(t) = G_j(t), \; j = 1, \ldots, M_4$$

$$A_1^{(k)}(t, \bar{t}) = [\sigma_n^{(k)+} + \sigma_n^{(k)-}] - i[\sigma_t^{(k)+} - \sigma_t^{(k)-}] -$$

$$2\Gamma - 2\bar{\Gamma} - 2\frac{dt}{d\bar{t}}\Gamma' + \sum_{j=1}^{K^*}\left[2\,\mathrm{Re}\frac{P_j + iQ_j}{\pi(1+k)}\frac{1}{t - z_j^*} -\right.$$

$$\left.\frac{dt}{d\bar{t}}\left(k\frac{P_j - iQ_j}{\pi(1+k)}\frac{1}{t - z_j^*} + \frac{P_j + iQ_j}{\pi(1+k)}\frac{\bar{t} - \bar{z}_j^*}{(t - z_j^*)^2}\right)\right] +$$

$$i\sum_{j=1}^{K^{**}}\frac{M_j}{\pi}\frac{dt}{d\bar{t}}\frac{1}{(t - z_j^{**})^2}$$

Taking account of the boundary coditions (2.5) and again the Sokhotskii-Plemelj formula for $\Phi_0(z)$ and $\Psi_0(z)$ as $z \to t \in \gamma_k^*$, we obtain

$$\frac{1}{\pi i}\oint_{\gamma_k^*}\frac{\varphi_k^*(\tau)}{\tau - t}\,d\tau - \frac{1}{\pi i}\oint_{\gamma_k^*}\frac{\overline{\varphi_k^*(\tau)}}{\bar{\tau} - \bar{t}}\,d\bar{\tau} - \tag{3.2}$$

$$\frac{dt}{d\bar{t}}\left[\frac{1}{\pi i}\oint_{\gamma_k^*}\frac{\overline{\varphi_k^*(\tau)}}{\tau - t}\,d\bar{\tau} + \frac{1}{\pi i}\oint_{\gamma_k^*}\frac{\bar{\tau} - \bar{t}}{(\tau - t)^2}\varphi_k^*(\tau)\,d\tau\right] +$$

$$\sum_{\substack{j=1 \\ j \neq k + M_1}}^{M}\frac{1}{\pi i}\int_{L_j^*}[K_{21}^{(j)}(\tau, t)\,\nu_j(\tau) + K_{22}^{(j)}(\tau, t)\,A(\tau, \bar{\tau})\,\overline{\nu_j(\tau)}]\,d\tau =$$

$$A_2^{(k)}(t, \bar{t}) - \frac{dt}{d\bar{t}}\left[\sum_{j=1}^{M_2}\frac{1}{\pi i}\oint_{\gamma_j^*}\frac{q_j^*(\tau)}{\tau - t}\,d\bar{\tau} + \sum_{j=1}^{M_1}\frac{1}{\pi i}\int_{l_j}\frac{q_1^{(j)}(\tau)}{\tau - t}\,d\bar{\tau}\right]$$

$$t \in \gamma_k^*, \quad (k = 1, \ldots, M_2)$$

Here

$$K_{21}^{(j)}(\tau, t) = \frac{1}{\tau - t}\left(1 - \frac{dt}{d\bar{t}}\frac{\bar{\tau} - \bar{t}}{\tau - t}\right), \quad j = 1, \ldots, M$$

$$K_{22}^{(j)}(\tau, t) = \begin{cases} -\dfrac{1}{\bar{\tau} - \bar{t}}\left(1 + \dfrac{dt}{d\bar{t}}\dfrac{\bar{\tau} - \bar{t}}{\tau - t}\right), & j = 1, \ldots, M_{12}, M_{123} + 1, \ldots, M \\ -\dfrac{1}{\bar{\tau} - \bar{t}}\left(1 - k\dfrac{dt}{d\bar{t}}\dfrac{\bar{\tau} - \bar{t}}{\tau - t}\right), & j = M_{12} + 1, \ldots, M_{123} \end{cases}$$

$$A_2^{(k)}(t, \bar{t}) = q_k^*(t) - \Gamma - \bar{\Gamma} - \frac{dt}{d\bar{t}}\Gamma' + \sum_{j=1}^{K^*}\left[\mathrm{Re}\frac{P_j + iQ_j}{\pi(1+k)}\frac{1}{t - z_j^*} -\right.$$

$$\left.\frac{dt}{d\bar{t}}\left(k\frac{P_j - iQ_j}{2\pi(1+k)}\frac{1}{t - z_j^*} + \frac{P_j + iQ_j}{2\pi(1+k)}\frac{\bar{t} - \bar{z}_j^*}{(t - z_j^*)^2}\right)\right] +$$

$$i\sum_{j=1}^{K^{**}}\frac{M_j}{2\pi}\frac{dt}{d\bar{t}}\frac{1}{(t - z_j^{**})^2}$$

If the boundary values $\Phi_0(z)$ and $\Psi_0(z)$ are replaced in conditions (2.7) as $z \to t \in L_k$, then by using the Sokhotskii-Plemelj formula we arrive at the following system of SIE:

$$i(k+1)\,h\overline{\mu_k(t)} + \frac{E^{(k)}S^{(k)}}{E}\frac{d}{dt}\left\{\mathrm{Re}\left[\frac{3 - \nu - k(1 + \nu)}{2}\mu_k(t) +\right.\right. \tag{3.3}$$

$$(1 - \nu)\frac{1}{\pi i}\int_{L_k}\frac{\mu_k(\tau)}{\tau - t}\,d\tau - (1 + \nu)\frac{dt}{d\bar{t}}\left\{\frac{k}{2\pi i}\int_{L_k}\frac{\overline{\mu_k(\tau)}}{\tau - t}\,d\bar{\tau} -\right.$$

$$\frac{1}{2\pi i}\int\limits_{L_k}\frac{\bar\tau-\bar t}{(\tau-t)^2}\mu_k(\tau)\,d\tau\Big\}\Big]\Big\}+\sum_{\substack{j=1\\ j\neq k+M_{11}}}^{M}\frac{1}{\pi i}\int\limits_{L_j^*}[K_{31}^{(j)}(\tau,t)\nu_j(\tau)+$$

$$K_{32}^{(j)}(\tau,t)A(\tau,\bar\tau)\overline{\nu_j(\tau)}]\,d\tau=A_3^{(k)}(t,\bar t)+$$

$$\frac{E^{(k)}S^{(k)}}{E}\frac{1+\nu}{2}\frac{d}{dt}\operatorname{Re}\Big\{\frac{dt}{\overline{dt}}\sum_{j=1}^{M_1}\frac{1}{\pi i}\int\limits_{l_j}\frac{\overline{q_1^{(j)}(\tau)}}{\tau-t}\overline{d\tau}+$$

$$\sum_{j=1}^{M_2}\frac{1}{\pi i}\oint\limits_{\gamma_j^*}\frac{\overline{q_j^*(\tau)}}{\tau-t}\overline{d\tau}\Big)\Big\},\quad t\in L_k\quad(k=1,\ldots,M_3)$$

Here

$$K_{31}^{(j)}(\tau,t)=\frac{1}{\tau-t}\Big(1-\nu+\frac{1+\nu}{2}\frac{dt}{\overline{dt}}\frac{\bar\tau-\bar t}{\tau-t}\Big),\quad j=1,\ldots,M$$

$$K_{32}^{(j)}(\tau,t)=\begin{cases}\dfrac{1+\nu}{2}\dfrac{dt}{\overline{dt}}\dfrac{1}{\bar\tau-\bar t}, & j=1,\ldots,M_{12},M_{123}+1,\ldots,M\\ -\dfrac{1+\nu}{2}k\dfrac{dt}{\overline{dt}}\dfrac{1}{\bar\tau-\bar t}, & j=M_{12}+1,\ldots,M_{123}\end{cases}$$

$$A_3^{(k)}(t,\bar t)=\frac{E^{(k)}S^{(k)}}{E}\frac{1+\nu}{2}\frac{d}{dt}\operatorname{Re}\Big\{\sum_{j=1}^{K^*}\Big\{\frac{2(1-\nu)}{1+\nu}\frac{P_j+iQ_j}{\pi(1+k)}\frac{1}{t-z_j^*}+$$

$$\frac{dt}{\overline{dt}}\Big(k\frac{P_j-iQ_j}{\pi(1+k)}\frac{1}{\bar t-\bar z_j^*}+\frac{P_j+iQ_j}{\pi(1+k)}\frac{\bar t-\bar z_j^*}{(t-z_j^*)^2}\Big)\Big\}-i\sum_{j=1}^{K^{**}}\frac{M_j}{\pi}\frac{1}{(t-z_j^{**})^2}\Big\}$$

Finally, substituting the boundary values $\Phi_0(z)$ and $\Psi_0(z)$ as $z\to t\in\gamma_k$ into (2.9) and using the Sokhotskii-Plemelj formula, we obtain yet another system of SIE

$$-(k_k+1+\Gamma_0^{(k)}+\Gamma_0^{(k)}k)\overline{G_k(t)}+\frac{1-\Gamma_0^{(k)}}{\pi i}\oint\limits_{\gamma_k}\frac{\overline{G_k(\tau)}}{\tau-t}\overline{d\tau}+\tag{3.4}$$

$$\frac{k_k-\Gamma_0^{(k)}k}{\pi i}\oint\limits_{\gamma_k}\frac{\overline{G_k(\tau)}}{\bar\tau-\bar t}\overline{d\tau}-(1-\Gamma_0^{(k)})\frac{dt}{\overline{dt}}\times$$

$$\Big\{\frac{1}{\pi i}\oint\limits_{\gamma_k}\frac{\overline{G_k(\tau)}}{\tau-t}\overline{d\tau}+\frac{1}{\pi i}\oint\limits_{\gamma_k}\frac{\bar\tau-\bar t}{(\tau-t)^2}G_k(\tau)\,d\tau\Big\}+$$

$$\sum_{\substack{j=1\\ j\neq k+M_{123}}}^{M}\frac{1}{\pi i}\int\limits_{L_j^*}[K_{41}^{(j)}(\tau,t)\nu_j(\tau)+K_{42}^{(j)}(\tau,t)A(\tau,\bar\tau)\overline{\nu_j(\tau)}]\,d\tau=$$

$$A_4^{(k)}(t,\bar t)-\frac{1-\Gamma_0^{(k)}}{\pi i}\frac{dt}{\overline{dt}}\Big\{\sum_{j=1}^{M_1}\int\limits_{l_j}\frac{\overline{q_1^{(j)}(\tau)}}{\tau-t}\overline{d\tau}+\sum_{j=1}^{M_2}\oint\limits_{\gamma_j^*}\frac{\overline{q_1^*(\tau)}}{\tau-t}\overline{d\tau}\Big\},$$

$$t\in\gamma_k\quad(k=1,\ldots,M_4)$$

Here

$$K_{41}^{(j)}(\tau,t)=\frac{1-\Gamma_0^{(k)}}{\tau-t}\Big(1-\frac{dt}{\overline{dt}}\frac{\bar\tau-\bar t}{\tau-t}\Big),\quad j=1,\ldots,M$$

$$K_{42}^{(j)}(\tau,t)=$$

$$\begin{cases}\dfrac{1}{\bar\tau-\bar t}\Big[k_k-\Gamma_0^{(k)}-(1-\Gamma_0^{(k)})\dfrac{dt}{\overline{dt}}\dfrac{\bar\tau-\bar t}{\tau-t}\Big], & j=1,\ldots,M_{12},\ M_{123}+1,\ldots,M\\ \dfrac{1}{\bar\tau-\bar t}\Big[k_k-\Gamma_0^{(k)}+k(1-\Gamma_0^{(k)})\dfrac{dt}{\overline{dt}}\dfrac{\bar\tau-\bar t}{\tau-t}\Big], & j=M_{12}+1,\ldots,M_{123}\end{cases}$$

$$A_4^{(k)}(t,\bar t)=2g_0^{(k)}(t)+2(\Gamma_0^{(k)}-1)\Gamma-2(\Gamma_0^{(k)}k-k_k)\bar\Gamma+$$

$$2\frac{dt}{\overline{dt}}(\Gamma_0^{(k)}-1)\Gamma'-\Gamma_0^{(k)}\sum_{j=1}^{K^*}\frac{P_j+iQ_j}{\pi(1+k)}\frac{1}{t-z_j^*}+$$

$$\Gamma_0^{(k)}k\sum_{j=1}^{K^*}\frac{P_j-iQ_j}{\pi(1+k)}\frac{1}{\bar t-\bar z_j^*}+\frac{dt}{\overline{dt}}\Big[\Gamma_0^{(k)}\Big[\sum_{j=1}^{K^*}\frac{k(P_j-iQ_j)}{\pi(1+k)}\times$$

$$\frac{1}{t-z_j^*}+\frac{P_j+iQ_j}{\pi(1+k)}\frac{\bar t-\bar z_j^*}{(t-z_j^*)^2}-i\sum_{j=1}^{K^{**}}\frac{M_j}{\pi}\frac{1}{(t-z_j^{**})^2}\Big]\Big]$$

To solve the systems of SIE (3.1)-(3.4) constructed, to which the generalized problem for a multiconnected domain reduces, we present expressions for the densities on the appropriate contours, taking the boundary conditions expressed in terms of the complex potentials into account

$$\varphi_j(t) = \frac{\overline{q_1^{(j)}(t)}}{k+1} + g_j(t), \quad t \in l_j \quad (j = 1, \ldots, M_1) \tag{3.5}$$

Here

$$g_j(t) = \frac{2\mu}{k+1} \frac{d}{dt} \{(u_j^+(t) - u_j^-(t)) + i(v_j^+(t) - v_j^-(t))\}|_{l_j} \tag{3.6}$$

$$\varphi_j^*(t) = \frac{1}{k+1} \overline{q_j^*(t)} + \frac{2\mu}{k+1} \frac{d}{dt}(u_j^-(t) + iv_j^-(t)), \quad t \in \gamma_j^* \quad (j = 1, \ldots, M_2)$$

$$\mu_j(t) = \frac{i(\sigma_t^+ - \sigma_t^-)}{k+1}, \quad t \in L_j \quad (j = 1, \ldots, M_3) \tag{3.7}$$

$$G_j(t) = p_j(t)\left[\frac{1}{k_j+1} - \frac{1}{k+1}\right] - \frac{\overline{g_0^{(j)}(t)}}{k_j+1} + \tag{3.8}$$
$$\left(\frac{2\mu_j}{k_j+1} - \frac{2\mu}{k+1}\right) \frac{d}{dt}(u_j^-(t) + iv_j^-(t)), \quad t \in \gamma_j \quad (j = 1, \ldots, M_4)$$

where $p_j(t) = \sigma_n - i\sigma_t$ is the unknown contact stress on the appropriate inclusion boundary.

For the solution of the problem to be unique, the system of SIE must be supplemented by a uniqueness condition for the displacements around the respective contours $\{l_j, \gamma_j^*, L_j, \gamma_j\}$.

We will have for each crack

$$\int_{l_j} \varphi_j(t)\, dt = \frac{1}{k+1} \int_{l_j} \overline{q_1^{(j)}(t)}\, dt, \quad j = 1, \ldots, M_1 \tag{3.9}$$

We will have for each hole outline

$$\oint_{\gamma_j^*} \Phi_0^-(t)\, dt = \frac{1}{k+1} \oint_{\gamma_j^*} \overline{q_j^*(t)}\, dt, \quad j = 1, \ldots, M_2 \tag{3.10}$$

The uniqueness condition for the displacements is the following for a rectilinear contour where the stringer L_j is located

$$\int_{L_j} \mu_j(t)\, dt = \frac{P_2^{(j)} - P_1^{(j)}}{(k+1)h}, \quad j = 1, \ldots, M_3 \tag{3.11}$$

where $P_k^{(j)}$ $(k = 1, 2)$ is the tensile or compressive force directed along the stringer and applied to its ends.

The uniqueness condition for displacements along the contact contour γ_j will have the form

$$\oint_{\gamma_j} [(1 - \Gamma_0^{(j)}) p_j(t) - (k_j + 1) \overline{\Phi_0^+(t)} + \Gamma_0^{(j)}(k+1) \overline{\Phi_0^-(t)}]\, dt = \tag{3.12}$$
$$\oint_{\gamma_j} g_0^{(j)}(t)\, \overline{dt}, \quad j = 1, \ldots, M_4$$

4. Examples of computations. 1. Let a stiff circular washer S_1 of unit radius be soldered in a circular hole in an infinite plate S_2. The difference between the washer and hole radii is $\delta = 0.001$ m. The plate is weakened by a rectilinear unloaded crack whose length equals the washer diameter. The elastic moduli and Poisson's ratios of the washer and plate are $E_1 = 20.59 \times 10^{10}$ N/m^2, $\nu_1 = 0.28$ and $E_2 = 3.34 \times 10^9$ N/m^2, $\nu_2 = 0.33$.

Taking account of the general approach, we obtain the following system of equations for the problem of inclusions with a crack

$$\frac{1}{\pi i} \int_l \frac{g(\tau)}{\tau - t} d\tau - \frac{1}{\pi i} \int_l \frac{\overline{g(\tau)}}{\bar\tau - \bar t} \overline{d\tau} - \frac{\overline{dt}}{dt} \left[\frac{1}{\pi i} \int_l \frac{\overline{g(\tau)}}{\tau - t} \overline{d\tau} + \frac{1}{\pi i} \int_l \frac{\bar\tau - \bar t}{(\tau - t)^2} g(\tau)\, d\tau\right] + \tag{4.1}$$
$$\frac{1}{\pi i} \oint_\gamma \frac{G(\tau)}{\tau - t} d\tau - \frac{1}{\pi i} \oint_\gamma \frac{\overline{G(\tau)}}{\bar\tau - \bar t} \overline{d\tau} -$$

$$\frac{dt}{d\bar t}\left[\frac{1}{\pi i}\oint_\gamma \frac{\overline{G(\tau)}}{\tau - t}\,d\bar\tau + \frac{1}{\pi i}\oint_\gamma \frac{\bar\tau - \bar t}{(\tau - t)^2}G(\tau)\,d\tau\right] = 0, \quad t\in l$$

$$-(k_1+1+\Gamma_0+\Gamma_0k_2)\overline{G(t)} + \frac{1-\Gamma_0}{\pi i}\left(\oint_\gamma \frac{G(\tau)}{\tau - t}\,d\tau + \int_l \frac{g(\tau)}{\tau - t}\,d\tau +\right.$$

$$\frac{k_1-\Gamma_0k_2}{\pi i}\left[\oint_\gamma \frac{\overline{G(\tau)}}{\bar\tau - \bar t}\,d\bar\tau + \int_l \frac{\overline{g(\tau)}}{\bar\tau - \bar t}\,d\bar\tau\right] - (1-\Gamma_0)\frac{dt}{d\bar t}\left\{\frac{1}{\pi i}\oint_\gamma \frac{\overline{G(\tau)}}{\tau - t}\,d\bar\tau +\right.$$

$$\left.\frac{1}{\pi i}\oint_\gamma \frac{\bar\tau - \bar t}{(\tau - t)^2}G(\tau)\,d\tau + \frac{1}{\pi i}\int_l \frac{\overline{g(\tau)}}{\tau - t}\,d\bar\tau + \frac{1}{\pi i}\int_l \frac{\bar\tau - \bar t}{(\tau - t)^2}g(\tau)\,d\tau\right\} = 2g_0(t), \quad t\in\gamma$$

$$\int_l g(t)\,dt = 0$$

$$\oint_\gamma \left\{(1-\Gamma_0)\,p(t) - \frac{\Gamma_0k_2+\Gamma_0+k_1+1}{2}\,\overline{G(t)} - \right.$$

$$\left.\frac{\Gamma_0k_2+\Gamma_0-k_1-1}{2\pi i}\oint_\gamma \frac{\overline{G(\tau)}}{\bar\tau - \bar t}\,d\bar\tau - \frac{\Gamma_0(k_2+1)}{2\pi i}\int_l \frac{\overline{g(\tau)}}{\bar\tau - \bar t}\,d\bar\tau\right\}\overline{dt} = 0$$

Here

$$\Gamma_0 = \mu_1/\mu_2,\ g_0(t) = 2\mu_1\delta$$

$$G(t) = p(t)\left[\frac{1}{1+k_1} - \frac{1}{1+k_2}\right] - \frac{2\mu_2}{1+k_2}\,\delta$$

$$g(t) = g_1(t) + ig_2(t) = \frac{2\mu_2}{1+k_2}\frac{d}{dt}\{(u^+(t) - u^-(t)) + i(v^+(t) - v^-(t))\}$$

$$(g(t) = i\{(b-t)(t-a)\}^{-1/2}g^*(t), \quad t\in[a,b])$$

Applying numerical integration formulas for functions along the circumference of a circle /21/

$$G(t) \approx u_n(G,t) = \frac{1}{2n+1}\sum_{j=-n}^{n} G(\tau_j)\left(\frac{t}{\tau_j}\right)^{-n}[1-(t/\tau_j)^{2n+1}][1-(t/\tau_j)]^{-1} \tag{4.2}$$

$$G(\tau_j) = u_n(G,\tau_j),\ \tau_j = \exp[2\pi(2n+1)^{-1}j]$$

$$\frac{1}{\pi i}\oint_\gamma \frac{G(\tau)}{\tau - t}\,d\tau \approx \frac{1}{\pi i}\oint_\gamma \frac{u_n(G,\tau)}{\tau - t}\,d\tau =$$

$$\frac{1}{2n+1}\sum_{j=-n}^{n} G(\tau_j)\left(1 + \frac{2i\sin[n(\vartheta-\vartheta_j)/2]\sin[(n+1)(\vartheta-\vartheta_j)/2]}{\sin[(\vartheta-\vartheta_j)/2]}\right)$$

$$\frac{1}{2\pi}\oint_\gamma G(\tau)\,d\vartheta = \frac{1}{2n+1}\sum_{j=1}^{2n+1} G(\tau_j)$$

and numerical integration formulas by the Lobatto-Chebyshev method for $g^*(t)$ /22, 23/, the values of the densities $g^*(t)$ and $G(t)$ can be determined at interpolation points on the integration contours. The Lobatto-Chebyshev method provides the possibility of determining at once the function $g^*(t)$ at the crack ends $[a, b]$, which means also the stress intensity factors at angular points /14/

$$K_{\mathrm I} = \left(\frac{2\pi}{\rho}\right)^{1/2}\left[\cos\frac{\psi+\vartheta}{2}\,g_1^*(b) + \sin\frac{\psi+\vartheta}{2}\,g_2^*(b)\right] \tag{4.3}$$

$$K_{\mathrm{II}} = \left(\frac{2\pi}{\rho}\right)^{1/2}\left[\sin\frac{\psi+\vartheta}{2}\,g_1^*(b) - \cos\frac{\psi+\vartheta}{2}\,g_2^*(b)\right]$$

$$\psi = \arg(b-a),\ \rho = |b-a|$$

where ϑ is the angle between the crack and the x axis.

This same problem was examined in /16/ as a special case of a contact problem for two bodies without friction, one of which has the crack.

The change in the stress intensity factors $K_{\mathrm I}$ and K_{II} is shown in Fig.2 as a function of the crack disposition relative to the stiff inclusion, while the change in the contact stress $p(t)$ at the point A is shown in Fig.3 as the crack approaches the contact boundary.

2. Let an infinite plate we weakened on one side by a rectilinear unloaded crack and on the other be reinforced by a stringer at whose ends tensile forces P are applied. The problem reduces to the following system of equations

$$\frac{1}{\pi i}\int_l \frac{g(\tau)}{\tau-t}\,d\tau - \frac{1}{\pi i}\int_l \frac{\overline{g(\tau)}}{\bar\tau-\bar t}\,d\bar\tau - \frac{dt}{d\bar t}\left[\frac{1}{\pi i}\int_l \frac{\overline{g(\tau)}}{\tau-t}\,d\bar\tau + \frac{1}{\pi i}\int_l \frac{\bar\tau-\bar t}{(\tau-t)^2}\,g(\tau)\,d\tau\right] + \tag{4.4}$$

$$\frac{1}{\pi i}\int_L \frac{\mu(\tau)}{\tau-t}\,d\tau - \frac{1}{\pi i}\int_L \frac{\overline{\mu(\tau)}}{\bar\tau-\bar t}\,d\bar\tau - \frac{dt}{d\bar t}\left[-\frac{k}{\pi i}\int_L \frac{\overline{\mu(\tau)}}{\tau-t}\,d\bar\tau + \right.$$

$$\left.\frac{1}{\pi i}\int_L \frac{\bar\tau-\bar t}{(\tau-t)^2}\,\mu(\tau)\,d\tau\right] = A_1(t,\bar t), \quad t\in l, \quad l=[a,b],$$

$$i(k+1)h\overline{\mu(t)} + \frac{E_0S_0}{E}\frac{d}{dt}\left\{\mathrm{Re}\left[\frac{3-\nu-k(1+\nu)}{2}\mu(t) + \right.\right.$$

$$(1-\nu)\left(\frac{1}{\pi i}\int_L \frac{\mu(\tau)}{\tau-t}\,d\tau + \frac{1}{\pi i}\int_l \frac{g(\tau)}{\tau-t}\,d\tau\right) -$$

$$(1+\nu)\frac{dt}{d\bar t}\left[\frac{k}{2\pi i}\int_L \frac{\overline{\mu(\tau)}}{\tau-t}\,d\bar\tau - \frac{1}{2\pi i}\int_L \frac{\bar\tau-\bar t}{(\tau-t)^2}\,\mu(\tau)\,d\tau - \right.$$

$$\left.\left.\left.\frac{1}{2\pi i}\int_l \frac{\overline{g(\tau)}}{\tau-t}\,d\bar\tau - \frac{1}{2\pi i}\int_l \frac{\bar\tau-\bar t}{(\tau-t)^2}\,g(\tau)\,d\tau\right]\right]\right\} = A_2(t,\bar t), \quad t\in L, \quad L=[-L_0, L_0]$$

$$\int_l g(t)\,dt = 0, \qquad \int_L \mu(t)\,dt = 0$$

Here $A_1(t,\bar t)$ and $A_2(t,\bar t)$ are expressions that are on the right-hand sieds of (3.1) and (3.3), where the forces $P_j + iQ_j$ and the moments M_j are applied at the points

$$z^*_{1,4} = z^{**}_{1,4} = L_0 \pm ic, \qquad z^*_{2,3} = z^{**}_{2,3} = -L_0 \pm ic$$

and

$$2P_j = \begin{cases} P, & j=1.4 \\ -P, & j=2.3 \end{cases} \quad Q_j = 0, \quad j=1,\ldots,4; \quad 2M_j = \begin{cases} Pc, & j=1.3 \\ -Pc, & j=2.4 \end{cases}$$

while c is a comparatively small quantity.

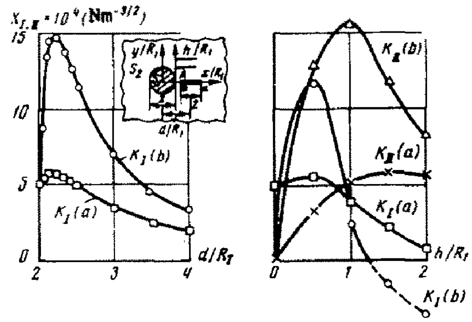

Fig.2

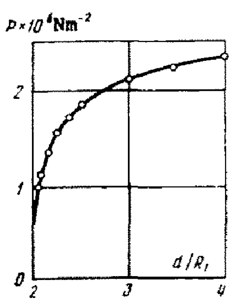

Fig.3

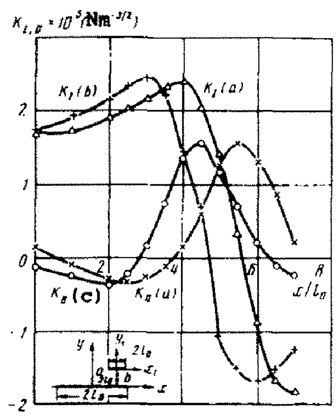

Fig.4

Since the singularity at the edge of the stringer is the same as for a crack, i.e.,

$$\mu(t) = i\{(-L_0 - t)(t - L_0)\}^{-1/2} \mu^*(t),\ t \in [-L_0, L_0]$$

we apply the Lobatto-Chebyshev method /15/ to solve system (4.4). After having solved the algebraic system of equations for the densities $g^*(t)$ and $\mu^*(t)$ at the interpolation points on the respective integration contours, the stress intensity factors K_I and K_{II} can be found at the crack edge by using (4.3).

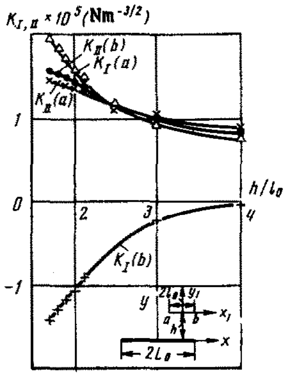

Fig.5

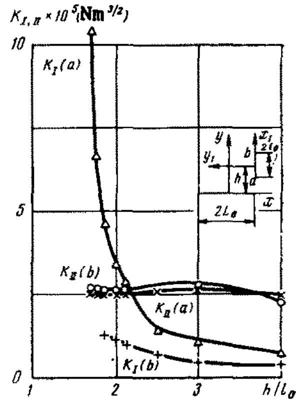

Fig.6

The change in the stress intensity factors K_I and K_{II} at the crack edge ($2l_0 = 0.02$ m is shown in Figs.4-6 for the case of an aluminium plate as a function of the crack location relative to a steel stringer ($2L_0 = 0.1$ m) at whose ends a 16 times greater force is applied than the magnitude of the force at infinity ($N_1 = 9.81 \times 10^4$ N/m^2, $N_2 = 0$).

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CONTINUAL-DISCRETE MODELLING OF A MULTICOMPONENT LAMINAR BODY BY USING A SYSTEM OF TWO-DIMENSIONAL CONTINUA*

T.A. PRIBYLEVA

A method is considered for constructing continual-discrete models of multicomponent layered bodies by using a system consisting of an arbitrary number of two-dimensional continua with finite intervals between them. Consistency relationships are presented for the fundamental kinematic, deformation, and dynamic parameters which enable rheological relationships to be obtained for the body as a whole taking the properties and nature of the interaction of the individual components into account. An example of the modelling of a thin laminar elastic body is examined. Methods for modelling a biological membrane are discussed.

Physical objects exist for which a direct description is impossible by methods of the mechanics of three-dimensional continuous media, or is insufficiently effective because the physical properties of the object are discrete in one of the directions, i.e., the requirements for the continuity hypothesis /1/ are not satisfied in this direction. The object here posssesses fairly continuous properties in the other two directions and allows of a continual description.

Among the discrete objects in the transverse direction is the shell of a live cell, a biological membrane, say, consisting of several layers of macromolecules where the individual layers include molecules of different species. Moreover, a broad class of laminar and stratified bodies exists, whose properties in the transverse direction can possibly be described by a discrete set of parameters.

In a number of papers (/2-7/, for example) the concept has been introduced of a two-dimensional continuum (a material surface possessing mass) that is characterized by appropriate kinematic, dynamic, and energy parameters. The ideal of modelling multicomponent laminar bodies by using systems of two-dimensional continua /8/ is

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